ON THE PRINCIPLE OF REDUCTION

PMM Vol. 31, No. 6, 1967, pp. 1086-1089

S.I. GORSHIN
(Alma-Ata)

(*Received August 13, 1967*)

1. Let us consider the following perturbed system of differential equations:

$$\frac{dy}{dt} = Y(t, x, y) + Q(t, x, y), \qquad \frac{dx}{dt} = X(t, x, y) + R(t, x, y) \tag{1.1}$$

where t is a real independent variable, while x and y are the required functions of t defined on a full, linear normed space E. Functions $X(t, x, y)$ and $Y(t, x, y)$ are given over a region

$$t \geqslant 0, \qquad \|x\| \leqslant H, \qquad \|y\| \leqslant H \tag{1.2}$$

of the space E and become zero when $x = y = \theta$. The functions $Q(t, x, y)$ and $R(t, x, y)$ are some unknown functions belonging to the space E; they characterize the persistent perturbations, and generally do not become zero when $x = y = \theta$ and satisfy, in the region (1.2), the conditions

$$\| Q(t, x, y) \| \leqslant \rho, \qquad \| R(t, x, y) \| \leqslant \rho \tag{1.3}$$

where $\rho > 0$ is a sufficiently small number.

We shall assume that the right-hand sides of (1.1) satisfy, in the region (1.2), the following conditions:

1) Functions $X(t, x, y)$, $Y(t, x, y)$, $Q(t, x, y)$ and $R(t, x, y)$ are single-valued and continuous in t,

2) For any two points (t, x', y') and (t, x'', y'') the following inequality holds:

$$\| U(t, x', y',) - U(t, x'', y'') \| \leqslant \alpha(t)\, \Delta u \quad (\Delta u = \max\, [\| x' - x'' \|, \| y' - y'' \|])$$

where $\alpha(t)$ is a real, bounded continuous function when $t \geqslant 0$ and $U(t, x, y)$ denotes any one of the functions listed in (1).

We shall call the variable y – critical, and x – fundamental. The solution $x = y = \theta$ of the system (1.1) without perturbations shall, in the following, be called the null solution. Let $x = z(t)$ be a continuous function in E satisfying the condition $\|z(t)\| \leqslant H$, and let us consider the following Eq.

$$dy \,/\, dt = Y(t, z(t), y) + Q(t, z(t), y) \tag{1.4}$$

obtained from the first Eq. of (1.1) by a substitution $x = z(t)$.

Together with (1.4), we shall consider an unperturbed Eq.

$$dy \,/\, dt = Y(t, z(t), y) \tag{1.5}$$

Definition 1.1. We shall say that solutions of (1.5) are stable under persistent perturbations when the fundamental variable is sufficiently small numerically, if, for any given number $\varepsilon > 0$ ($\varepsilon < H$), for any initial value $t = t_0 \geqslant 0$ and under any indicated choice of $x = z(t)$, there exist two other numbers $r = r(\varepsilon, t_0) > 0$ and $\rho = \rho(\varepsilon, t_0) > 0$ such, that as soon as

$$\| y(t_0) \| \leqslant r, \qquad \| z(t_0) \| \leqslant r$$

the inequality

$$\| z(t) \| \leqslant \varepsilon$$

will also hold for any $t > t_0$ for which

$$\| y(t) \| < \varepsilon$$

under any perturbations $Q(t, x, y)$ satisfying the condition (1.3). Here $y(t)$ is the solution of (1.4) passing through the point (t_0, y_0).

If the numbers $r > 0$ and $\rho > 0$ are independent of the initial value of $t = t_0 \geqslant 0$, then the solutions of (1.5) shall be called uniformly stable under persistent perturbations when the numerical value of the fundamental variable is sufficiently small.

Definition 1.2. We shall say that solutions of (1.5) are unstable under persistent perturbations when the numerical value of the fundamental variable is sufficiently small if, for any $t = t_0 \geqslant 0$ and any sufficiently small number $\rho > 0$, there exists such point (t_0, y_0) with the value of $\|y_0\| > 0$ arbitrarily small and such a perturbation $Q(t, x, y)$ satisfying (1.3), that under any indicated choice of $x = z(t)$, a solution belonging to solutions of (1.4) passing through this point can always be found such, that it will at some $t > t_0$ satisfy the inequality

$$\| y(t) \| \geqslant \varepsilon$$

where $\varepsilon > 0$ ($\varepsilon < H$) is a constant which is independent of the choice of $x = z(t)$ of the given point (t_0, y_0) and of the choice of perturbations $Q(t, x, y)$.

We then have the following Theorems:

Theorem 1.1. If a perturbed system of differential equations is such that solutions of the first equation are stable under persistent perturbations when the numerical value of the fundamental variable is sufficiently small, while the solutions of the second equation are stable under persistent perturbations when the numerical value of the critical variable is sufficiently small, then the null solution is stable under persistent perturbations.

Proof. Let the number $\varepsilon > 0$ ($\varepsilon < H$) and the initial value $t = t_0 \geqslant 0$ be given. Assume that $x = x(t)$, $y = y(t)$ is a solution of the perturbed system (1.1) and that this solution passes through the point (t_0, x_0, y_0). Then the theorem implies that numbers $r_1 = r_1(\varepsilon, t_0) > 0$ and $\rho_1 = \rho_1(\varepsilon, t_0) > 0$ can be found such, that as soon as

$$\| x(t_0) \| \leqslant r \qquad \| y(t_0) \| \leqslant r_1$$

the inequality $\|y(t)\| < \varepsilon$ will also hold for all $t \geqslant t_0$ for which

$$\| x(t) \| \leqslant \varepsilon, \qquad \| Q(t, x, y) \| \leqslant \rho_1$$

Moreover we can also find numbers $r_2 = r_2(\varepsilon, t_0) > 0$ and $\rho_2 = \rho_2(\varepsilon, t_0) > 0$ such that as soon as $\|x(t_0)\| \leqslant r_2$ and $\|y(t_0)\| \leqslant r_2$, the inequality $\|x(t)\| < \varepsilon$ will also hold for all $t > t_0$ for which $\|y(t)\| \leqslant \varepsilon$ and $\|R(t, x, y)\| \leqslant \rho_2$.

If we put $r = \min(r_1, r_2)$ and $\rho = \min(\rho_1, \rho_2)$ then as soon as $\|x(t_0)\| \leqslant r$ and $\|y(t_0)\| \leqslant r$, we shall have, for all $t > t_0$ for which $\|x(t)\| \leqslant \varepsilon$ and $\|Q(t, x, y)\| \leqslant \rho$,

$$\| y(t) \| < \varepsilon \tag{1.6}$$

while for all $t > t_0$ for which $\|y(t)\| \leqslant \varepsilon$ and $\|R(t, x, y)\| \leqslant \rho$, we shall have

$$\| x(t) \| < \varepsilon \tag{1.7}$$

From this it follows that the inequalities (1.6) and (1.7) cannot simultaneously become equalities, hence neither of them can become an equality. Therefore (1.6) and (1.7) hold for all $t \geqslant t_0 \geqslant 0$ i.e. the null solution is stable under persistent perturbations.

Theorem 1.2. If a perturbed system of differential equations is such that solutions of the first equation are unstable under persistent perturbations when the numerical value of the fundamental variable is sufficiently small, then the null solution is unstable under persistent perturbations.

The Theorem is obvious, and we can consider the second equation of the system instead of its first equation

2. Let us assume that the function $X(t, x, y)$ has the form

$$X(t, x, y) = P(t, x) + N(t, y) + L(t, x, y) \tag{2.1}$$

where $P(t, x)$ is continuous in t, linear in x, i.e.

$$P(t, \alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 P(t, x_1) + \alpha_2 P(t, x_2)$$

and satisfies the condition

$$\| P(t, x) \| \leqslant \| x \| p(t)$$

where α_1 and α_2 denote any real numbers and $p(t)$ is a real, continuous and bounded function. Functions $N(t, y)$ and $L(t, x, y)$ satisfy the inequalities

$$\| N(t, y) \| \leqslant \| y \| \gamma(\| y \|), \qquad \| L(t, x, y) \| \leqslant \| x \| \delta(\| x \|, \| y \|)$$

where $\gamma(\| y \|) \to 0$ when $\| y \| \to 0$, and $\delta(\| x \|, \| y \|) \to 0$ when $\| x \| + \| y \| \to 0$.

Let us now assume that the linear Eq.

$$dx / dt = P(t, x)$$

is such, that its bounded solution $x = f(t, t_0, x_0)$ passing through the point (t_0, x_0) satisfies, for all $t \geqslant t_0 \geqslant 0$, the inequality

$$\| f(t, t_0, x_0) \| \leqslant \| x_0 \| B \exp[-\alpha(t - t_0)] \tag{2.2}$$

where $B \geqslant 1$ and $\alpha > 0$ are some constants independent of t_0 and x_0.

We have now the following theorem:

Theorem 2.1. If a perturbed system of differential equations is such that solutions of the first equation are stable under persistent perturbations when the numerical value of the fundamental variable is sufficiently small, while the function X has the form (2.1) and satisfies the indicated conditions, then the null solution is stable under persistent perturbations.

Proof. Let the number $\varepsilon > 0$ $(\varepsilon < H)$ and the initial value $t = t_0 \geqslant 0$ be given and let $r_1 = r_1(\varepsilon, t_0)$ and $\rho_1 = \rho_1(\varepsilon, t_0)$ be the numbers defined by (1.5). Assume that

$$r = \min\left(r_1, \frac{\varepsilon}{5B}\right), \qquad \rho = \min\left(\rho_1, \frac{\alpha\varepsilon}{5B}\right)$$

Let us now suppose that $x = x(t)$ and $y = y(t)$ is a solution of (1.1), which passes through the point (t_0, x_0, y_0) and satisfies the condition

$$\| x(t_0) \| \leqslant r, \qquad \| y(t_0) \| \leqslant r$$

Then, by conditions of the Theorem it follows that for all $t \geqslant t_0 \geqslant 0$ for whic

$$\| x(t) \| \leqslant \varepsilon \tag{2.3}$$

holds, we also have

$$\| y(t) \| < \varepsilon \tag{2.4}$$

under any perturbations $Q(t, x, y)$ satisfying the conditions (1.3).

Let us consider a segment $[t_0, t]$ on which the inequality (2.3) and hence (2.4) holds. Then the function $x = x(t)$ appearing in the solution of (1.1) will satisfy [2]

$$x(t) = f(t, t_0, x_0) + \int_{t_0}^{t} f[t, \tau, N(\tau, y(\tau))]\, d\tau +$$

$$+ \int_{t_0}^{t} f[t, \tau, L(\tau, x(\tau), y(\tau))]\, d\tau + \int_{t_0}^{t} f[t, \tau, R(\tau, x(\tau), y(\tau))]\, d\tau$$

Hence

$$\| x(t) \| \leqslant rBe^{[-\alpha(t-t_0)]} + \int_{t_0}^{t} Be^{[-\alpha(t-\tau)]} \varepsilon\gamma(\varepsilon)\, d\tau +$$

$$+ \int_{t_0}^{t} Be^{[-\alpha(t-\tau)]}\, \varepsilon\delta(\varepsilon, \varepsilon)\, d\tau + \int_{t_0}^{t} Be^{[-\alpha(t-\tau)]}\, \rho\, d\tau$$

This implies that the inequality

$$\|x(t)\| \leqslant rB + \varepsilon B\alpha^{-1}\gamma(\varepsilon) + \varepsilon B\alpha^{-1}\delta(\varepsilon, \varepsilon) + B\alpha^{-1}\rho \leqslant {}^4/_5\, \varepsilon$$

holds on the segment $[t_0, t]$, provided that the number $\varepsilon > 0$ is chosen small enough to satisfy

$$B\alpha^{-1}\gamma(\varepsilon) \leqslant {}^1/_5, \qquad B\alpha^{-1}\delta(\varepsilon, \varepsilon) \leqslant {}^1/_5$$

and that perturbations $R(t, x, y)$ satisfy the condition (1.3).

Consequently, if the inequality (2.3) holds on $[t_0, t]$, then a stronger inequality

$$\|x(t)\| \leqslant {}^4/_5\, \varepsilon$$

also holds. Therefore (2.3) and (2.4) will hold for any $t \geqslant t_0 \geqslant 0$ and any perturbations satisfying (1.3), i.e. the null solution is stable under persistent perturbations.

BIBLIOGRAPHY

1. Persidskii, K.P., Some critical cases of countable systems. Izv. Akad. Nauk KazSSR, Ser. mat. mekh., Vol. 5, 1951.
2. Persidskii, K.P., On stability of solutions of a countable system of differential equations. Izv. Akad. Nauk KazSSR, Ser. mat. mekh., Vol. 2, 1948.
3. Iataev, M., Analysis of a critical case of stability. Izv. Akad. Nauk KazSSR, Ser. fiz.-mat., No. 129, 1954.
4. Gorshin, S.I., On stability of solutions of equations in a linear normed space under persistent perturbations. Izv. Akad. Nauk KazSSR, Ser. fiz.-mat., Vol. 3, 1965.

Translated by L.K.